1. (a) Since the coordinates of the points P, Q and R are $(4,1,-1)$, $(3,3,5)$ and $(1,0,2c)$, respectively, the vectors $\overrightarrow{QR}$ and $\overrightarrow{PR}$ are given by

$\overrightarrow{QR} = -2\vec{i} - 3\vec{j} + (2c-5)\vec{k}$ *(M1)(A1)*

$\overrightarrow{PR} = -3\vec{i} - \vec{j} + (2c+1)\vec{k}$ *(M1)(A1)*

$\overrightarrow{QR}$ is perpendicular to $\overrightarrow{PR}$ if and only if $\overrightarrow{QR} \cdot \overrightarrow{PR} = 0$ *i.e.* $6+3+(2c-5)(2c-1)=0$ *(M1)*

$\Rightarrow \quad 4c^2 - 8c + 4 = 0$

$\Rightarrow \quad (c-1)^2 = 0$ *(M1)*

$\Rightarrow \quad c = 1$ *(A1)*

(b) $\overrightarrow{PR} = -3\vec{i} - \vec{j} + 3\vec{k}$, $\overrightarrow{PS} = -3\vec{i} + 3\vec{k}$ *(M1)(M1)*

$$\overrightarrow{PS} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 0 & 3 \\ -3 & -1 & 3 \end{vmatrix}$$ *(M1)*

$= 3\vec{i} + 3\vec{k}$ *(A1)*

(c) The parametric equation of a line l which passes through the point $(3,3,5)$ and is parallel to the vector $\overrightarrow{PR}$ is given by

$\vec{r} = (3\bar{\mathbf{i}} + 3\bar{\mathbf{j}} + 5\bar{\mathbf{k}}) + t(-3\bar{\mathbf{i}} - \bar{\mathbf{j}} + 3\bar{\mathbf{k}})$ *(M1)(M1)*

$= 3(1-3t)\vec{i} + (3-t)\vec{j} + (5+3t)\vec{k} \quad (-\infty < t < \infty)$ *(A1)*

Note:	If $-\infty < t < \infty$ is not mentioned, do not penalise. Also note that some candidates may give the parametric equation of the line in the form $x = 3(1-t), y = 3-t, z = (5+3t), -\infty < t < \infty$

continued...

Question 1 continued

(d) Let P_1 and P_2 be points on the line l corresponding to $t=0$ and $t=1$, respectively. Hence, for $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,
$x = 3(1-t)$, $y = (3-t)$ and $z = 5+3t$.
Putting $t=0$ and $t=1$, we get the coordinates of points P_1 and P_2 as $(3,3,5)$ and $(0,2,8)$, respectively.

Vectors $\overrightarrow{SP_1}$ and $\overrightarrow{SP_2}$ are given by $\overrightarrow{SP_1} = 2\vec{i} + 2\vec{j} + 3\vec{k}$ and $\overrightarrow{SP_2} = -\vec{i} + \vec{j} + 6\vec{k}$. *(M1)*

A vector perpendicular to both $\overrightarrow{SP_1}$ and $\overrightarrow{SP_2}$ is $\overrightarrow{SP_1} \times \overrightarrow{SP_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 3 \\ -1 & 1 & 6 \end{vmatrix}$ *(M1)*

$= 9\vec{i} - 15\vec{j} + 4\vec{k}$ *(A1)*

Let $T(x,y,z)$ be any point of the plane π. Since $S = (1,1,2)$, $\overrightarrow{ST} = (x-1)\vec{i} + (y-1)\vec{j} + (z-2)\vec{k}$ is a vector in π. Hence $\overrightarrow{TS} \times \vec{n} = 0$

i.e. $9(x-1) - 15(y-1) + 4(z-2) = 0 \Rightarrow 9x - 15y + 4z - 2 = 0$ *(A1)*

OR

$\overrightarrow{QS} = (1-3)\vec{i} + (1-3)\vec{j} + (2-5)\vec{k}$ *(M1)*

$= -2\vec{i} - 2\vec{j} - 3\vec{k}$ *(M1)*

The equation of the plane containing the line l and passing through the point S is determined by l and the vector $\overrightarrow{SQ}$. Hence, the equation is:

$$\vec{r} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix}$$

(M1)(A1)

(e) Shortest distance $\dfrac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|} = \dfrac{\begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -15 \\ 4 \end{pmatrix}}{\sqrt{322}}$ *(M1)(M1)*

$= \dfrac{15}{\sqrt{322}}$ *(M1)(A1)*

OR

The distance of P from π is:

$$\frac{|9(4) - 15(1) + 4(-1) - 2|}{\sqrt{9^2 + 15^2 + 4^2}}$$ *(M1)(A1)*

$= \dfrac{15}{\sqrt{322}}$ *(M1)(A1)*

Note:	Accept 0.836 (3 s.f.)

2. (i) Let the arithmetic sequence be written as $a, a+d, a+2d, \ldots$

Then $\dfrac{a+4d}{a+11d} = \dfrac{6}{13}$ *(M1)*

So $13a + 52d = 6a + 66d \Rightarrow 7a = 14d$ *(M1)*

$\Rightarrow a = 2d$. *(A1)*

Since each term is positive, both a and d are positive. We are given $a(a+2d) = 32$, setting $a = 2d$, we get $2d(2d+2d) = 8d^2 = 32$. *(M1)*

$\Rightarrow d = \pm 2$. *(A1)*

Hence, $d = 2$ and $a = 4$ and sum to 100 terms of this sequence is

$\dfrac{100}{2}\{(2)(4) + (100-1)2\}$. *(M1)*

$= 10300$ *(A1)*

(ii) (a) Since ω is a complex number which satisfies $\omega^3 - 1 = 0$, $\omega \neq 1$. Hence,

$1 + \omega + \omega^2 = \dfrac{1-\omega^3}{1-\omega} = 0$. *(M1)(A1)*

(b) $(\omega x + \omega^2 y)(\omega^2 x + \omega y) = \omega^3 x^2 + \omega^4 yx + \omega^2 xy + \omega^3 y^2$. *(M1)*

Using $\omega^3 = 1$ and $\omega^4 = \omega$, we get, *(M1)*

$(\omega x + \omega^2 y)(\omega^2 x + \omega y) = (x^2 + y^2) + (\omega^2 + \omega)xy$ *(M1)*

$= x^2 + y^2 - xy$, (Since $1 + \omega + \omega^2 = 0$) *(A1)*

(iii) Let $S(n)$ be the statement: $2^{2n} - 3n - 1$ is divisible by 9.

Since $2^2 - 3 - 1 = 0$, $S(1)$ is true. *(C1)*

Assume as the induction hypothesis $S(k)$ *i.e.* $2^{2k} - 3k - 1$ is divisible by 9. *(C1)*

We shall show that $S(k+1)$ is true.

$S(k+1) = 2^{2(k+1)} - 3(k+1) - 1$ *(M1)*

$= 4(2^{2k}) - 3k - 4$ *(M1)*

$= 4(2^{2k} - 3k - 1) + 9k$ *(A1)*

By the induction hypothesis $2^{2k} - 3k - 1$ is divisible by 9. Since $9k$ is also divisible by 9, $S(k+1)$ is true. *(M1)*

Thus, by mathematical induction $S(n)$ is true, $n = 1, 2, \ldots$ *(R1)*

3. (i) Let D be the event that the patient has the disease and S be the event that the new blood test shows that the patient has the disease. Let D' be the complement of D, *i.e.* the patient does not have the disease.
Now the given probabilities can be written as
$p(S|D) = 0.99$, $p(D) = 0.0001$, $p(S|D') = 0.05$. *(A1)(A1)(A1)*

Since the blood test shows that the patient has the disease, we are required to find $p(D|S)$.
By Bayes' theorem,

$$p(D|S) = \frac{p(S|D)p(D)}{p(S|D)p(D) + p(S|D')p(D')}$$ *(M1)*

$$= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.05)(1-0.0001)}$$ *(M1)*

$$= 0.001976\ldots = 0.00198 \text{ (3 s.f.)}$$ *(A1)*

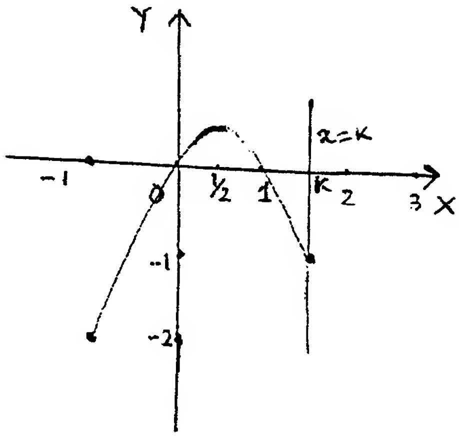
Note: Some candidates may use a tree diagram. Please check the steps and award marks accordingly.

(ii) Let n = number of chips = 1000,
p = the probability that a randomly chosen chip is defective = 0.02.
Hence, the mean $np = (1000)(0.02) = 20$ and the
variance $= np(1-p) = (1000)(0.02)(0.98) = 19.6$. *(A1)(A1)*
Suppose X is the normal random variable that approximates the binomial distribution.
The $X \approx N(20, 19.6)$. *(M1)(M1)*

Thus $p(19.5 \le X \le 30.5) = p\left(\frac{19.5-20}{\sqrt{19.6}} \le Z \le \frac{30.5-20}{\sqrt{19.6}}\right)$ *(M1)(M1)*
$= p(-0.11 \le Z \le 2.37)$
$= 0.5349$ *(A1)*

Notes: Some may write $p(-0.113 \le Z \le 2.372) = 0.5362$. Award full marks. Accept answers correct to 3 s.f.

4. (i) (a)



Notes:	Award *(A1)* for the correct shape
	(A1) for both end points
	(A1) for the line $x = k$

(b) Required area $= \int_0^1 (x - x^2)\,dx - \int_1^k (x - x^2)\,dx$ *(M1)*

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 - \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_1^k$$ *(A1)*

$$= \frac{1}{6} - \frac{k^2}{2} + \frac{k^3}{3} + \frac{1}{6} = \frac{1}{3} + \frac{k^3}{3} - \frac{k^2}{2}$$ *(M1)*

$$= \frac{1}{6}(2 + 2k^3 - 3k^2)$$ *(A1)*

continued...

Question 4 continued

(ii) (a) $g(t) = \frac{\ln t}{\sqrt{t}}$. So $g'(t) = \frac{2 - \ln t}{2t^{3/2}}$. *(M1)(A1)*

Hence, $g'(t) \ge 0$ when $2 \ge \ln t$ or $\ln t \le 2$ or $t \le e^2$. *(M1)*

Since the domain of $g(t)$ is $\{t : t > 0\}$, $g'(t) \ge 0$ when $0 < t \le e^2$. *(A1)*

(b) Since $g'(t) = \frac{2 - \ln t}{2t^{3/2}}$, $g''(t) = \frac{-2\sqrt{t} - 3\sqrt{t}(2 - \ln t)}{4t^3}$ *(M2)*

$$= -\frac{\sqrt{t}[8 - 3\ln t]}{4t^3}$$ *(A1)*

Hence $g''(t) > 0$ when $8 - 3\ln t < 0$ *i.e.* $t > e^{8/3}$.
Similarly, $g''(t) < 0$ when $0 < t < e^{8/3}$. *(M1)*
Hence there is a point of inflexion when $t = e^{8/3}$. *(A1)*

(c) $g''(t) = 0$ when $t = 0$ or $8 = 3\ln t$.
Since, the domain of g is $\{t : t > 0\}$, $g''(t) = 0$ when $t = e^{8/3}$. *(M1)*
Since $g''(t) > 0$ when $t > e^{8/3}$ and $g''(t) < 0$ when $t < e^{8/3}$, *(M1)*
$\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$ is the point of inflexion. The required value of t is $e^{8/3}$. *(A1)*

Note: Award *(A1)* for evaluating t as $e^{8/3}$.

(d) $g'(t) = 0$ when $\ln t = 2$ or $t = e^2$. *(M1)*
Also $g''(e^2) = -\frac{\sqrt{e^2}[8 - \ln e^2]}{4(e^2)^3} = -\frac{6e}{4e^6} < 0$. *(M1)*
Hence $t^* = e^2$ *(A1)*

(e) At $(t^*, g(t^*))$ the tangent is horizontal. *(M1)*
So the normal at the point $(t^*, g(t^*))$ is the line $t = t^*$. *(M1)*
Thus, it meets the t axis at the point $t = t^* = e^2$ and hence the point is $(e^2, 0)$. *(A1)*

5. (i) (a) S is the group of permutations of $\{1,2,3\}$ under the composition of permutation. Since $3! = 6$, order of $S = 6$. *(M1)* *(R1)*

(b) Members of S are $p_0 = \begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$, $p_1 = \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}$, $p_2 = \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}$, *(AG)*

$$p_3 = \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix},\ p_4 = \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix},\ p_5 = \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}.$$ *(A2)*

> **Note:** Award *(A2)* for 3 correct permutations;
> *(A1)* for 2 correct permutations;
> *(A0)* for 1 correct permutation.

Since $p_3 \circ p_4 = \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}$

and $p_4 \circ p_3 = \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}$, *(M1)*

$p_3 \circ p_4 \neq p_4 \circ p_3$ *(R1)*

> **Note:** There are other possibilities to show that the group is not Abelian.

(c) $p_1^2 = \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix} = p_2$

$p_1^3 = \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix} = p_0$. *(M1)*

(Note that p_0 is the identity of the group S.)

Hence $\{p_0, p_1, p_2\}$ form a cyclic group of order 3 under the binary operation of composition of permutations. *(R1)*

> **Note:** Some candidates may write $\{p_0, p_1, p_2\}$ is a subgroup of order 3, (award *(A1)*), and write the following table, (award *(R1)*):

$\circ$	p_0	p_1	p_2
p_0	p_0	p_1	p_2
p_1	p_1	p_2	p_0
p_2	p_2	p_0	p_1

continued...

Question 5 continued

(ii) (a) Let $A = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$.

Let $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ be two elements of A. Then

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{pmatrix}$$ ***(M1)***

$= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, if we write $\alpha = ac - bd, \beta = ad + bc$. ***(R1)***

Since α, β are real numbers A is closed under matrix multiplication.
Associativity follows from the fact that matrix multiplication is an associative binary operation on the collection of all 2×2 matrices and the fact that A is closed with respect to the binary operation of multiplication of matrices. ***(R1)***

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in A$ and is the identity since $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ ***(M1)(M1)***

for any element $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ of A. ***(R1)***

Since $a \neq 0$, the determinant $\begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2 + b^2 \neq 0$. ***(M1)***

Hence the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ has an inverse, which is $\frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ ***(R1)(A1)***

since $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\left(\frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ***(R1)***

> **Note:** Some candidates may use $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to find the inverse of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Award marks as appropriate.

continued…

Question 5 (ii) continued

(b) $M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

M contains the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *(A1)*

The following results show that the set M is closed with respect to matrix multiplication as the binary operation. *(R1)*

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *(M2)*

Since the product of each element with itself is the identity, every element of M is its own inverse. Hence M is a subgroup of A. *(R1)*

(c) The order of the group M is 4. If it had a subgroup of order 3, then by the corollary of the Lagrange's theorem, 3 must be a divisor of 4. *(M1)*
Since it is not true, M can not have a subgroup of order 3. *(R1)*

(iii) (a) Let $(G, \circ)$ and $(H, \bullet)$ be two groups. They are said to be isomorphic if there exists a one-to-one transformation $f : G \to H$ which is surjective (onto) with the property that for all $x, y \in G$, $f(x \circ y) = f(x) \bullet f(y)$. *(C1)* *(C1)*

Note:	Some candidates may say that the groups $(G, \circ)$ and $(H, \bullet)$ are isomorphic if they have the same Cayley table (or group table). In that case award *(C1)*.

(b) Since $f : G \to H$, $f(x) \in H$ for some $x \in G$.
Since e' is the identity element in H,
$e' \bullet f(x) = f(x) = f(x \circ e) = f(e) \bullet f(x)$. *(M1)* *(M1)(A1)*
By the right cancellation law, $e' = f(e)$. *(R1)*

continued...

Question 5 (iii) continued

(c) Suppose $G = \langle a \rangle$, the cyclic group generated by a, *i.e.* n is the smallest positive integer such that $a^n = e$, the identity in G. ***(C1)***
Let $f : G \to H$ be an isomorphism. Let $f(a) = b \in H$.
$f(a^2) = f(a \circ a) = f(a) \bullet f(a) = (f(a))^2$. In general $f(a^n) = (f(a))^n$. ***(M1)(A1)***
By (iii) (b) $(f(a))^n = e'$, the identity in H. Hence $b^n = e'$ and consequently H is a cyclic group of order n with generator b. ***(R1)***

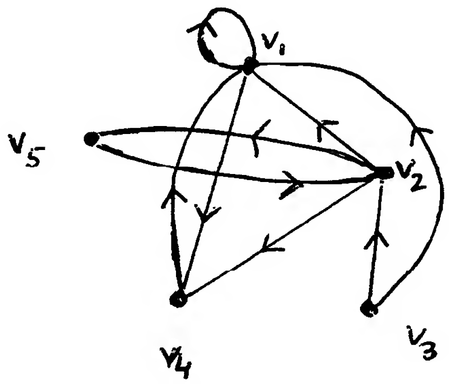
(d) $\mathbb{Z}_4 = \{[0],[1],[2],[3]\}$. Let $\oplus_4$ be the binary operation. ***(M1)***
$[1] \oplus_4 [1] = [2], [1] \oplus_4 [2] = [3], [1] \oplus [3] = [0]$ ***(M1)(A1)***
Hence $[1]$ is a generator of the group $(\mathbb{Z}_4, \oplus_4)$ and $\mathbb{Z}_4$ is cyclic.
Since each element of M is of order 2, M is not cyclic. ***(M1)***
Hence $\mathbb{Z}_4$ is not isomorphic to M. ***(R1)***

6. (i) (a) If G is a directed graph with n vertices, then an adjacency matrix of G is an $n \times n$ matrix whose i, j entry is a number different from zero if there is a directed edge from the vertex v_i to the vertex v_j, and zero if there is no directed edge connecting v_i and v_j. *(C2)*

(b) The sum of the entries in row i of the adjacency matrix of the directed graph G is equal to the outdegree of the vertex v_i of G, *i.e.* the number of directed edges from vertex v_i. *(C2)*

(c) The sum of the entries in column j equals the indegree of the vertex j *i.e.* the number of directed edges to vertex v_j. *(C2)*

(ii) (a)



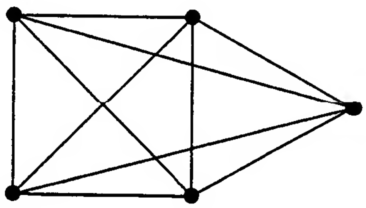
Graph H *(A3)*

Note:	Award *(A3)* for 6 correct, *(A2)* for 5 correct; *(A1)* for 4 correct, *(A0)* for 3 or fewer correct

(b) Since the element in the second row and the fourth column of A^2 is 1, there is only one path of length 2 from v_2 to v_4. *(M1)(R1)*

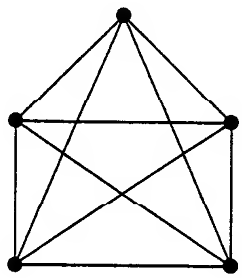
(iii) (a) (i) A complete graph κ_5 is a graph on 5 vertices where every vertex is connected to every other vertex. *(A2)*

Example:



κ_5

or



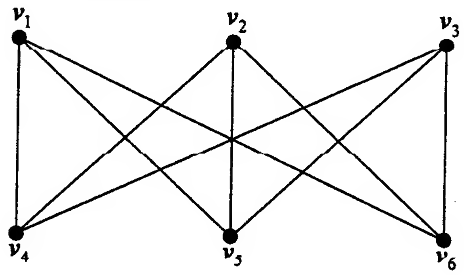
κ_5 *(A1)*

continued...

Question 6 (iii) continued

(ii) In $\kappa_{3,3}$ vertices are divided into two disjoint sets A and B, each having three vertices such that no two vertices in A are adjacent and no two vertices in B are adjacent, but every vertex of A is adjacent to every vertex of B. *(A1)*

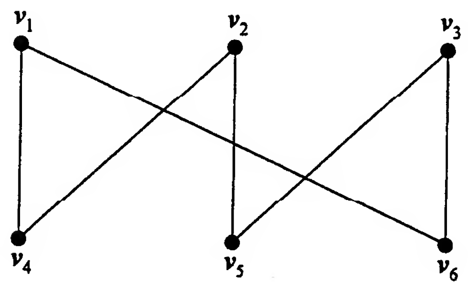
Example:



(A1)

$\kappa_{3,3}$

(b) To find a Hamiltonian circuit in $\kappa_{3,3}$, we show that there is a circuit which goes through each vertex exactly once. Such a circuit is shown below: *(M1)*



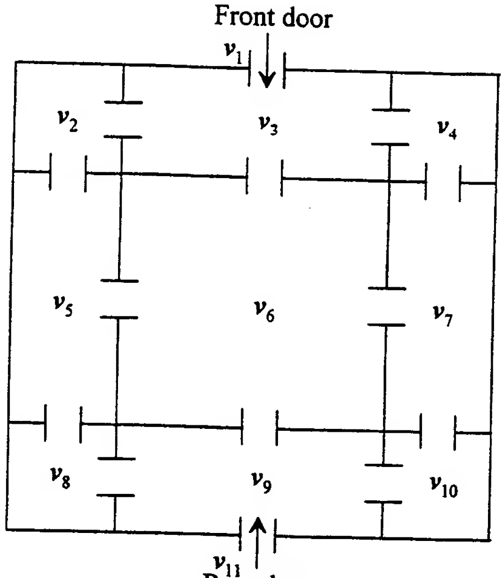
(M1)(A1)

Note:	Please check the circuit drawn by the candidate and award *(M2)(A1)* for a correct answer. Also note that some candidates may not show the Hamiltonian circuit but mention that each vertex of $\kappa_{3,3}$ has degree $3 = \frac{6}{2}$. Hence there is a Hamiltonian circuit. In this case award *(M2)(A1)*.

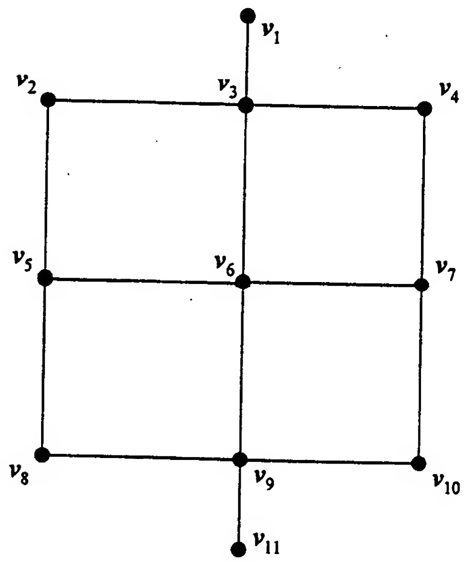
continued...

Question 6 continued

(iv)



Let the front door, rear door and each room of the house be labelled as shown. We take each of these as a vertex, and draw edges when they are connected by a door. The graph of the floor plan is: *(M1)*



(M1)(A1)

The degree sequence for this graph $v_1, v_2, \ldots, v_{11}$ is $1, 2, 4, 2, 3, 4, 3, 2, 4, 2, 1$. *(M1)(A1)*

The degree of each vertex is not even and hence there is no Eulerian circuit. *(R1)*

Hence it is not possible to enter the house through the front door and exit at the rear door, travelling through the house going through each internal doorway exactly once. *(R1)*

continued...

Question 6 continued

(v) (a) The degree sequence yields the number of edges at each vertex and an isomorphism preserves adjacency of vertices. *(M1)*

Hence, if two groups are isomorphic and the degree sequences are not the same then the isomorphism can not preserve the adjacency of vertices contradicting our observation. *(M1)*

Thus an isomorphism must preserve degree sequences. *(R1)*

(b) The degree sequences of the graphs G and H are $2,3,3,3,3,4$ and $2,3,3,3,5,2$ respectively. *(M1)(M1)*

Since they are not the same, the two graphs are not isomorphic by (a). *(R1)*

(vi) We start with point A and write S as the set of vertices and T as the set of edges. The weights on each edge will be used in applying Prim's algorithm.

Initially, $S = \{A\}$, $T = \phi$. In each subsequent stage, we shall update S and T.

<u>Step 1:</u>	Add edge h:	So $S = \{A, D\}$,	$T = \{h\}$
<u>Step 2:</u>	Add edge e:	So $S = \{A, D, E\}$	$T = \{h, e\}$
<u>Step 3:</u>	Add edge d:	Then $S = \{A, D, E, F\}$	$T = \{h, e, d\}$
<u>Step 4:</u>	Add edge a:	Then $S = \{A, D, E, F, B\}$	$T = \{h, e, d, a\}$
<u>Step 5:</u>	Add edge i:	Then $S = \{A, D, E, F, B, G\}$	$T = \{h, e, d, a, i\}$
<u>Step 6:</u>	Add edge g:	Then $S = \{A, D, E, F, B, G, C\}$	$T = \{h, e, d, a, i, g\}$ *(M3)(A3)*

Note: Award *(M3)(A3)* for all 6 correct, *(M3)(A2)* for 5 correct; *(M2)(A2)* for 4 correct, *(M2)(A1)* for 3 correct; *(M1)(A1)* for 2 correct, *(M1)(A0)* for 1 correct.

Now S has all the vertices and the minimal spanning tree is obtained by going through the vertices $A \to D \to E \to F \to B \to G \to C$. *(M1)*

The weight of the edges in T is $5+3+5+7+5+6=31$ *(R1)*

7. (i) (a) Let X denote the number of flaws in one metre of the wire. Then $E(X) = 2.3$ flaws and $P(X = 2) = e^{-2.3}\dfrac{(2.3)^2}{2!}$ *(M1)(M1)*

$= 0.265$. *(A1)*

(b) Let Y denote the number of flaws in two metres of wire. Then Y has a Poisson distribution with mean $E(Y) = 2 \times 2.3 = 4.6$ flaws for 2 metres. *(M1)*

Hence, $P(Y \geq 1) = 1 - P(Y = 0) = 1 - e^{-4.6}$ *(M1)*

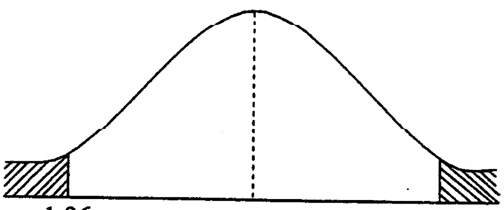
$= 0.9899$ *(A1)*

Note: Accept 0.990 (3 s.f.), or $1 - e^{-4.6}$.

(ii) Let $\bar{x}$ be the sample mean, μ be the population mean and σ be the population standard deviation.

Hence, $\sigma_{\bar{x}}$, the standard error is given by $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{4}{\sqrt{n}}$, where n is the sample size which is required. *(M1)(A1)*

The 95 % confidence interval of the population mean μ is given by $(\bar{x} - \mu) \pm 1.96\sigma_{\bar{x}}$. *(M1)(A1)*



Hence, we need to find n so that $(\sigma_{\bar{x}})(1.96) \leq 0.25$ *(M1)(R1)*

$$\Rightarrow \left(\frac{4}{\sqrt{n}}\right)(1.96) \leq 0.25$$

$$\Rightarrow \left(\frac{1}{\sqrt{n}}\right) \leq \frac{0.25}{4(1.96)} \Rightarrow \sqrt{n} \geq \frac{4(1.96)}{0.25} = 31.36$$ *(M1)(M1)*

Hence, $n \geq 983.44$ *(A1)*

The sample size required is 984. *(R1)*

Note: Award ***(M2)(R2)*** if candidates write: "We need to find n such that $\left(\frac{4}{\sqrt{n}}\right)(1.96) \leq 0.25$."

continued...

Question 7 continued

(iii) Let μ_1 and μ_2 be the weekly mean wage of science and humanities students respectively. The hypotheses to be tested are

$H_0: \mu_1 - \mu_2 = 0$ *(A1)*
$H_1: \mu_1 - \mu_2 \neq 0$ *(A1)*

It is a two-tailed test. The test statistic is $\bar{x}_1 - \bar{x}_2$. The two random variables are independent *i.e.* probability of selection of a science student is not affected by the selection of a humanity student.

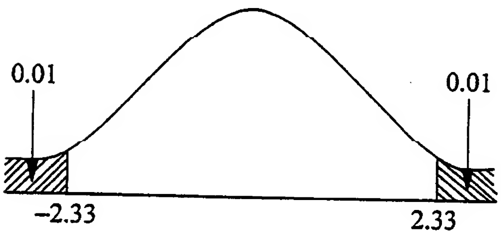
$$\mu_{\bar{x}_1-\bar{x}_2} = 0, \sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where σ_1, σ_2 are the population standard deviations of science and humanity students respectively. Since the samples are large one can approximate σ_1, σ_2 by s_1, s_2, respectively. The resulting estimated standard error is given by

$$s_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$ *(M2)*

$$= \sqrt{\frac{4^2}{100} + \frac{(4.5)^2}{200}}$$ *(A1)*

$$= 0.51\ldots$$ *(A1)*



From the normal distribution table we obtain the decision rule that if the computed value of $z > 2.33$ or $z < -2.33$ then we reject H_0. *(M1)(A1)*

$$\text{But } z = \frac{\bar{x}_1 - \bar{x}_2}{s_{\bar{x}_1-\bar{x}_2}}$$ *(M1)*

$$= \frac{\$\,120.5 - 115}{0.51}$$ *(M1)*

$$= 10.784 > 2.33$$ *(A1)*

So we reject H_0. *(R1)*

Thus, we conclude the wages of science and humanity students are not the same.

Note:	An alternative approach by calculating critical values for a rejection region is possible. Please check the working and award marks accordingly.

continued...

Question 7 continued

(iv) H_0: the cost of the automobile is independent of the number of complaints.
H_1: the cost of an automobile is not independent of the number of complaints. *(A1)*

The observed frequencies are first totalled, and then the expected frequencies under H_0 are calculated from the formula:

$$\text{expected frequency} = \frac{(\text{row total})(\text{column total})}{\text{grand total}}$$

Cost	Number of complaints			Total
	≤5	6–10	≥11	
≥ \$30001	**60**	**120**	**20**	200
\$15001 – \$30000	**138**	**276**	**46**	460
≤ \$15000	**102**	**204**	**34**	340
Total	300	600	100	1000

(A4)

Table of expected frequencies

Note: Award *(A4)* for 9 correct bold entries
(A3) for 7 or 8 correct bold entries
(A2) for 5 or 6 correct bold entries
(A1) for 4 correct bold entries
(A0) for 3 or less bold entries

$$\chi^2 = \sum\left\{\frac{(f_o - f_e)^2}{f_e}\right\}$$ *(M1)*

We shall calculate χ^2 from a table where we shall list $f_o, f_e, (f_o - f_e)^2, \frac{(f_o - f_e)^2}{f_e}$, where f_o is the observed number of complaints and f_e is the expected number of complaints.

continued...

Question 7 (iv) continued

	f_o	f_e	$f_o - f_e$	$(f_o - f_e)^2$	$\frac{(f_o - f_e)^2}{f_e}$
	100	60	40	1600	26.67
	90	120	−30	900	7.5
	10	20	−10	100	5.0
	150	138	12	144	1.04
	260	276	−16	256	0.93
	50	46	4	16	0.35
	50	102	52	2704	26.51
	250	204	46	2116	10.37
	40	34	6	36	1.06
Total	1000	1000			$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 79.43$

(A3)

Calculation of χ^2

v = number of degrees of freedom $= (3-1)(3-1) = 4$ *(M1)*

At the 5 % level of significance, $\chi^2_{0.05}$ with four degrees of freedom, $\chi^2_{0.05} = 9.49$ *(A1)*

Decision rule: If $\chi^2 > 9.49$, reject H_0. If $\chi^2 \le 9.49$, then accept H_0.
Our computed $\chi^2 = 79.43 > 9.49$ and hence according to our decision rule, we reject H_0.

Conclusion: The cost of the automobile is not independent of the number of complaints. *(R1)*

Note: Award *(M1)(R1)* for stating: $\chi^2_{0.05} = 9.49 < 79.43$, hence reject H_0.

8. (i) (a) Let $f(x) = x^3 - 3x - 5$. Hence $f(1) = -7 < 0$ and $f(3) = 13 > 0$. *(M1)*
Hence, by the intermediate value theorem there is one zero of $f(x) = x^3 - 3x - 5$ in the interval $1 \le x \le 3$. *(A1)*

(b) $f(x) = x^3 - 3x - 5$
$f'(x) = 3x^2 - 3$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3x_n - 5}{3x_n^2 - 3}$$ *(M1)(A1)*

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2	2.333333	**0.333333**
1	2.333333	2.280556	**0.052777**
2	2.280556	2.279020	**0.001536**
3	2.279020	2.279019	**1×10^{-6}**
4	2.279019	2.279019	**1×10^{-6}**

(A3)

Note: Award *(A3)* for 5 correct bold entries
(A2) for 4 correct bold entries
(A1) for 2 or 3 correct bold entries
(A0) for 0 or 1 correct bold entries

Since we need accuracy 10^{-5}, we need $\lvert x_n - x_{n+1} \rvert < 5\times10^{-6}$.
So $x = 2.27902$ is the solution of $f(x) \approx 0$. *(R1)*

(ii) (a) Formula for the trapezium rule with $n = 6$ gives,

$$\int_0^1 g(x)\,\mathrm{d}x = \frac{1}{12}[g(x_0) + 2g(x_1) + 2g(x_2) + 2g(x_3) + 2g(x_4) + 2g(x_5) + g(x_6)]$$ *(M2)*

$= 0.745$ *(A1)*

Formula for Simpson's rule gives,

$$\int_0^1 g(x)\,\mathrm{d}x = \frac{1}{18}[g(x_0) + 4g(x_1) + 2g(x_2) + 4g(x_3) + 2g(x_4) + 4g(x_5) + g(x_6)]$$ *(M2)*

$= 0.747$ *(A1)*

(b) $\lvert \text{Error by the Simpson's rule} \rvert \le \max\limits_{0\le x\le 1} \lvert g^{(4)}(x) \rvert \left(\frac{1}{180}\right)\left(\frac{1}{6}\right)^4$ *(M1)(A1)*

continued...

Question 8 (ii) continued

(c) Let $S_n(g)$ be the approximation to $\int_0^1 g(x)\mathrm{d}x$ by Simpson's rule over n intervals of equal length.

So $\left|\int_0^1 g(x)\mathrm{d}x - S_n(g)\right| \le \dfrac{6}{(180)n^4}$ *(M1)*

$= \dfrac{1}{30n^4}$, *(A1)*

Since $\left|g^{(4)}(x)\right| \le 6$ for $0 \le x \le 1$.

To be correct to 5 d.p., we need to find n so that *(M1)*

$\dfrac{1}{30n^4} \le 5\times 10^{-6}$ *(M1)*

i.e. $30n^4 \ge \dfrac{10^6}{5}$ or $n^4 \ge \dfrac{10^6}{150}$ or $n \ge \sqrt[4]{\dfrac{10^6}{150}} = 9.036$ *(M1)*

Hence, we need 10 intervals. *(A1)*

(iii) (a) If $g(x)$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$, then there exists a ξ in $a < \xi < b$ so that
$g(b) - g(a) = (b-a)g'(\xi), a < \xi < b$. *(A1)* *(A1)*

(b) By the mean value theorem $h(x) - h(0) = h'(c)(x-0)$, for $0 < c < 7$. *(M1)*

Therefore $|h(x) - h(0)| \le 10x$, for $0 \le x \le 7$, since $|h'(x)| \le 10$. *(A1)*

Thus $-70 + h(0) \le h(x) \le h(0) + 70$ *(M1)*

$\Rightarrow \quad -70 - 4 \le h(x) \le -4 + 70$

$\Rightarrow \quad -74 \le h(x) \le 66$

Hence $h(x) \ge -74$, for $0 \le x \le 7$. *(A1)*

(iv) (a) Let $u_k = \dfrac{k+1}{3^k}$.

Then $\lim\limits_{k\to\infty} \dfrac{u_{k+1}}{u_k} = \lim\limits_{k\to\infty} \dfrac{(k+2)}{3^{k+1}} \times \dfrac{3^k}{(k+1)} = \dfrac{1}{3} < 1$. *(M2)*

and hence the series converges by the ratio test. *(R1)*

continued...

Question 8 (iv) continued

(b) Let $f(x) = \dfrac{1}{x(\ln x)^3}$

Since $x(\ln x)^3$ is an increasing function of x for $x > 1$, $f(x)$ is a decreasing function of x for $x > 1$. *(M1)*

Further, $f(k) = \dfrac{1}{k(\ln k)^3}$

Hence, by the integral test, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges or diverges according to the convergence or divergence of $\displaystyle\int_2^{\infty} \frac{1\mathrm{d}x}{x(\ln x)^3} = \lim_{R\to\infty} \int_2^R \frac{\mathrm{d}x}{x(\ln x)^3}$ *(M1)*

$$= \lim_{R\to\infty} \left[\frac{(\ln x)^{-2}}{-2}\right]_2^R = \lim_{R\to\infty} \left\{-\frac{1}{2(\ln R)^2} + \frac{1}{2(\ln 2)^2}\right\} = \frac{1}{2(\ln 2)^2}$$ *(A1)*

So the series converges, by the integral test. *(R1)*

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2+1}$ is an alternating series.

When $f(x) = \dfrac{x}{x^2+1}$, $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} \le 0$, *(M1)(A1)*

when $x \ge 1$, the sequence $\left\{\dfrac{k}{k^2+1}\right\}$ is a decreasing sequence.

Also $\displaystyle\lim_{k\to\infty} \frac{k}{k^2+1} = \lim_{k\to\infty} \frac{1}{k + \frac{1}{k}} = 0.$ *(M1)(A1)*

Hence the series converges by the alternating series test. *(R1)*